

Steady-state quantities of interest of Markovian Queues

system type	M/M/1	M/M/m	M/M/ ∞	M/M/1/K
state space	\mathbb{N}_0	\mathbb{N}_0	\mathbb{N}_0	$\{0, 1, \dots, K\} \subset \mathbb{N}_0$
mean arrival rate λ_i	$\lambda \ (\forall i)$	$\lambda \ (\forall i)$	$\lambda \ (\forall i)$	$\begin{cases} \lambda & (i < K) \\ 0 & (i \geq K) \end{cases}$
mean service rate μ_i	$\mu \ (\forall i)$	$\begin{cases} i \mu & (i < m) \\ m\mu & (i \geq m) \end{cases}$	$i \mu \ (\forall i)$	$\mu \ (\forall i)$
average service time $\vartheta_s = E[\vartheta_s]$	$\frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu}$
ergodicity conditions	$\varrho = \frac{\lambda}{\mu} < 1$	$\varrho = \frac{\lambda}{m\mu} < 1$	$\forall \varrho = \frac{\lambda}{\mu}$	$\forall \varrho = \frac{\lambda}{\mu}$
steady-state (s.s.) probabilities $\pi_i \equiv \Pr(x(\infty) = i)$	$(1 - \varrho)\varrho^i \ (\forall i)$	(*)	$\frac{\varrho^i}{i!}e^{-\varrho} \ (\forall i)$	$\begin{cases} \varrho^i \frac{1 - \varrho}{1 - \varrho^{K+1}} & (i \leq K) \\ 0 & (i > K) \end{cases}$
idle probability $\pi_0 = \Pr(x(\infty) = 0)$	$(1 - \varrho)$	(*)	$e^{-\varrho}$	$\frac{1 - \varrho}{1 - \varrho^{K+1}}$
utilization factor (prop. of time the sys. is busy) $\hat{U} = \Pr(x(\infty) > 0) = 1 - \pi_0$	ϱ	$1 - \pi_0$	$1 - e^{-\varrho}$	$\varrho \frac{1 - \varrho^K}{1 - \varrho^{K+1}}$
single server utilization $\tilde{\varrho} = \bar{x}_s/m$	ϱ	ϱ	0	$\varrho \frac{1 - \varrho^K}{1 - \varrho^{K+1}}$
blocking probability (prob. a customer is rejected)	0	0	0	$\pi_K = \varrho^K \cdot \pi_0$
arrival rate within the resource $\lambda_{in} = \sum_{i=0}^{\infty} \lambda_i \pi_i$	λ	λ	λ	$\lambda - \lambda_{ab} = \lambda \frac{1 - \varrho^K}{1 - \varrho^{K+1}}$
abandonment rate $\lambda_{ab} = \lambda - \lambda_{in}$	0	0	0	$\lambda \cdot \pi_K = \lambda \frac{\varrho^K (1 - \varrho)}{1 - \varrho^{K+1}}$
throughput (productivity) $\eta = \hat{U} \cdot \mu \equiv \lambda_{in}$	λ	λ	λ	$\lambda \frac{1 - \varrho^K}{1 - \varrho^{K+1}}$
expected number of customer in the system at s.s. $\bar{x} = \sum_{i=0}^{\infty} i \cdot \pi_i \equiv \lambda_{in} \cdot \bar{\vartheta}$	$\frac{\varrho}{1 - \varrho}$	$m\varrho + \frac{m^m \varrho^{m+1}}{m!(1 - \varrho)^2} \pi_0$	ϱ	$\frac{\varrho(1 - (K+1)\varrho^K + K\varrho^{K+1})}{(1 - \varrho^{K+1})(1 - \varrho)}$
expected number of of busy servers at s.s. $\bar{x}_s = \lambda_{in} \cdot \bar{\vartheta}_s$	ϱ	$m\varrho$	ϱ	$\varrho \frac{1 - \varrho^K}{1 - \varrho^{K+1}}$
expected number of customers in the buffer at s.s. $\bar{x}_b = \bar{x} - \bar{x}_s \equiv \lambda_{in} \cdot \bar{\vartheta}_b$	$\frac{\varrho^2}{1 - \varrho}$	$\bar{x} - m\varrho$	0	$\frac{\varrho^2(1 - K\varrho^{K-1} + (K-1)\varrho^K)}{(1 - \varrho^{K+1})(1 - \varrho)}$
expected time spent in the system $\bar{\vartheta} = \bar{x}/\lambda_{in}$	$\frac{1}{\mu(1 - \varrho)}$	$\frac{\bar{x}}{\lambda}$	$\frac{1}{\mu}$	$\frac{(1 - (K+1)\varrho^K + K\varrho^{K+1})}{\mu(1 - \varrho^K)(1 - \varrho)}$
expected time spent in the buffer $\bar{\vartheta}_b = \bar{\vartheta} - \bar{\vartheta}_s \equiv \bar{x}_b/\lambda_{in}$	$\frac{\varrho}{\mu(1 - \varrho)}$	$\frac{\bar{x}}{\lambda} - \frac{1}{\mu}$	0	$\frac{\varrho(1 - K\varrho^{K-1} + (K-1)\varrho^K)}{\mu(1 - \varrho^K)(1 - \varrho)}$

(*) Stationary state probabilities of a M/M/m queue system:

$$\pi_{s,0} = \frac{1}{\left(\sum_{i=0}^{m-1} \frac{m^i \varrho^i}{i!}\right) + \frac{m^m}{m!} \sum_{i=m}^{\infty} \varrho^i} = \frac{1}{\left(\sum_{i=0}^{m-1} \frac{m^i \varrho^i}{i!}\right) + \frac{m^m}{m!} (1 - \varrho)}, \quad \begin{cases} \pi_{s,i} = \frac{m^i \varrho^i}{i!} \pi_{s,0} & (i < m) \\ \pi_{s,i} = \frac{m^m}{m!} \varrho^i \pi_{s,0} & (i \geq m) \end{cases}$$

Gordon-Newell Theorem: Consider an ergodic Gordon-Newell Network of v $M/M/m_i$ nodes, and n costumers within. Let $\mathbf{X} = (X_1, X_2, \dots, X_v)$ be a random vector, and X_i be the discrete random variable counting the number of costumers in the i -th node of the network. Let $N_{v,n}$ be the sample-space of all the possible arrangement of the n costumers in the network. Consider $\mathbf{x} = (x_1, x_2, \dots, x_v) \in N_{v,n}$. Then, the following steady-state joint probability takes the following product form

$$\Pr(\mathbf{X}(\infty) = \mathbf{x}) = \Pr(X_1(\infty) = x_1, X_2(\infty) = x_2, \dots, X_v(\infty) = x_v) = \kappa \cdot \prod_{i=1}^v \beta_i(x_i)$$

where κ is a normalization constant derived by the following constraint

$$\sum_{\forall (x_1, x_2, \dots, x_v) \in N_{v,n}} \kappa \cdot \prod_{i=1}^v \beta_i(x_i) = 1 \quad \rightarrow \quad \kappa = \frac{1}{\left(\sum_{\forall (x_1, x_2, \dots, x_v) \in N_{v,n}} \prod_{i=1}^v \beta_i(x_i) \right)}$$

whereas the functions $\beta_i(x_i)$ are selected in accordance with the next relations

$$\text{if the } i\text{-th node is a } M/M/1 : \beta_i(x_i) = \rho_i^{x_i} \quad \forall x_i \in [0, n]$$

$$\text{if the } i\text{-th node is a } M/M/m_i : \beta_i(x_i) = \begin{cases} \frac{\rho_i^{x_i}}{x_i!} & \text{if } x_i < m_i \\ \frac{\rho_i^{x_i}}{m_i! m_i^{x_i - m_i}} & \text{if } x_i \geq m_i \end{cases}$$

and $\rho_i = \lambda_i / \mu_i$, and λ_i is a solution of $\boldsymbol{\lambda} = \boldsymbol{\lambda} \cdot \mathbf{R}$, where $\mathbf{R} \in \mathbb{R}^{v \times v}$ is routing probability matrix of the network and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_v) \in \mathbb{R}_{\geq 0}^v$. ■

Markov's Inequality: If X is a nonnegative random variable and $a > 0$, then the probability that X is at least a is at most the expectation of X , $\mu = E[X]$, divided by a :

$$\Pr(X \geq a) \leq \frac{\mu}{a}$$

■

Chebyshev's Inequality: Let X (integrable) be a random variable with finite expected value μ and finite non-zero variance σ^2 . Then for any real number $a > 0$,

$$\Pr(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}.$$

■

Kolmogorov's Inequality: Let X_1, \dots, X_n be n i.i.d. r.v. with expected value $\mu = E[X_i]$ and variance $\text{Var}[X_i] < +\infty$ for $i = 1, \dots, n$. Let $S_k = \sum_{i=1}^k X_i$. Then for any real number $a > 0$,

$$\Pr\left(\max_{1 \leq k \leq n} |S_k - E[S_k]| \geq a\right) \leq \frac{1}{a^2} \cdot \text{Var}[S_k].$$

■

68-95-99.7 rule: In statistics, the 68-95-99.7 rule, also known as the empirical rule, is a shorthand used to remember the percentage of values that lie within a band around the mean in a normal distribution with a width of two, four and six standard deviations, respectively; more accurately, 68.27%, 95.45% and 99.73% of the values lie within one, two and three standard deviations of the mean, respectively. ■

Discrete distributions

distribution of $X = (S, p)$	Sample space	pmf	cdf	mean	variance	MGF
non - par. r.v. X	S	$p_X = \Pr(X = x_i)$	$P_X = \Pr(X \leq x)$	$\mu_X = E[X]$	$\sigma_X^2 = E[(X - \mu_X)^2]$	$M_X(z)$
	$\{x_0, x_1, \dots, x_n\}$	$\Pr(X = x_i)$	$\Pr(X \leq x)$	$\sum_{x_i \in X} x_i \cdot p_X(x_i)$	$\sum_{x_i \in X} (x_i - \mu_X)^2 \cdot p_X(x_i)$	$\sum_{k=0}^n p_X(x_k) \cdot z^{-k}$
$X \sim \text{Ber}(q), q \in [0, 1]$	$\{0, 1\}$	$\begin{cases} 1 - q & X = 0 \\ q & X = 1 \end{cases}$	$\begin{cases} 0 & X < 0 \\ 1 - q & X < 1 \\ 1 & X \geq 1 \end{cases}$	q	$q(1 - q)$	$(1 - q) + qz^{-1}$
$X \sim \text{Geo}(q), q \in [0, 1]$	$k \in \{0, 1, 2, \dots\}$	$(1 - q)^{k-1}q$	$1 - (1 - q)^k$	$\frac{1}{q}$	$\frac{1 - q}{q^2}$	$\frac{q}{z - (1 - q)}$
$X \sim \text{Bin}(n, q), q \in [0, 1]$	$\{1, 2, \dots, n\}$	$\binom{n}{k} (1 - q)^{n-k} q^k$	$\sum_{i=0}^{\lfloor k \rfloor} q^i (1 - q)^{n-i}$	nq	$nq(1 - q)$	$((1 - q) + qz^{-1})^n$
$X \sim \text{Pois}(\lambda), \lambda \in \mathbb{R}_+$	$\{0, 1, 2, \dots\}$	$e^{-\lambda} \frac{\lambda^i}{k!}$	$e^{-\lambda} \sum_{i=0}^{\lfloor k \rfloor} \frac{\lambda^i}{i!}$	λ	λ	—
$X \sim \text{Uni}(a, b), a, b \in \mathbb{Z}, a \leq b$	$k \in \{a, a + 1, \dots, b - 1, b\}$	$\frac{1}{b - a + 1}$	$\frac{\lfloor k \rfloor - a + 1}{b - a + 1}$	$\frac{a + b}{2}$	$\frac{(b - a + 1)^2 + 1}{12}$	$\frac{z^{-a} - z^{-b-1}}{(b - a + 1)(1 - z^{-1})}$

Continuous distributions

distribution of	Sample space	pdf	cdf	mean	variance	MGF
$X = (S, p)$	S	$f_X = \Pr(X = x_i)$	$F_X = \Pr(X \leq x)$	$\mu_X = E[X]$	$\sigma_X^2 = E[(X - \mu_X)^2]$	$M_X(s)$
$X \sim \text{Exp}(\lambda)$	$\mathbb{R}_{\geq 0}$	$1 - e^{-\lambda x}$	$\lambda \cdot e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{s + \lambda}$
$X \sim \text{Erl}(k, \lambda), k \in \mathbb{N}_{>0}, \lambda \in \mathbb{R}_+$	$\mathbb{R}_{\geq 0}$	$\frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}$	$1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\lambda x} (\lambda x)^n$	$\frac{k}{\lambda}$	$\frac{k}{\lambda^2}$	$\frac{\lambda^k}{(s + \lambda)^k}$
$X \sim \text{Norm}(\mu, \sigma^2), \mu_X \in \mathbb{R}, \sigma \in \mathbb{R}_{>0}$	\mathbb{R}	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\frac{1}{2} \left[1 + \text{erf} \left(\frac{x-\mu}{\sigma\sqrt{2}} \right) \right]$	μ	σ^2	—
$X \sim \text{Uni}(a, b), a, b \in \mathbb{Z}, a \leq b$	$[a, b]$	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	—

Probability formulas

Probability definition	$\Pr(A) = \lim_{N \rightarrow \infty} \frac{N_A}{N}$
Addition rule	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
Conditional probability	$\Pr(A B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \Pr(B) > 0$
Law of total probability	$\Pr(A) = \sum_{\forall i} \Pr(B_i) \cdot \Pr(A B_i)$
Bayes' Theorem	$\Pr(B_j A) = \frac{\Pr(A \cap B_j)}{\sum_{\forall i} \Pr(B_i) \cdot \Pr(A B_i)}$

List of mathematical series

Proposition: Let $k, m, n \in \mathbb{N}$ and $\rho \in \mathbb{R}$ the following identities holds:

- *Arithmetic progression:*

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}$$

- *Finite geometric progression:*

$$\sum_{k=m}^n \rho^k = \begin{cases} \frac{1-\rho^{n-m+1}}{1-\rho} \rho^m & \text{if } \rho \neq 1 \\ n - m + 1 & \text{if } \rho = 1 \end{cases}$$

- *Infinite geometric progression:*

$$\sum_{k=m}^{\infty} \rho^k = \frac{1}{1-\rho} \rho^m \quad \text{if } |\rho| < 1$$

- *Weighted infinite geometric progression:*

$$\sum_{k=m}^{\infty} k \rho^k = \frac{m + \rho - m\rho}{(1-\rho)^2} \rho^m \quad \text{if } |\rho| < 1$$

- *Exponential series:*

$$\sum_{k=0}^{\infty} \frac{\rho^k}{k!} = e^\rho$$

■

Proof: The first identity is an elementary result of math. Legend says that it was proved also by the young J.C.F Gauss. Its proof is as follows. Let us first note that the progression

$$S_n = \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n$$

can be rewritten as $S_n = a_n + a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1$.

Thus, one has that because of $(a_1 + a_n) = (a_i + a_{n-i+1})$ for all $i = 1, 2, \dots, n$, then $2S_n = n(a_1 + a_n)$. By letting $a_1 = 1$, and $a_n = n$ one derives the first identity.

The second identity can be proved by noticing that $(1 - \rho^{n+1}) = (1 - \rho)(1 + \rho + \rho^2 + \cdots + \rho^n)$, while taking into account that

$$\sum_{k=m}^n \rho^k = \rho^m \sum_{k=0}^{n-m} \rho^k.$$

The third identity follows by the evaluating the limit of the second relation as $n \rightarrow \infty$. Let us further note that, in this case, if $m = 0$, one get that $\sum_{k=0}^{\infty} \rho^k = 1/(1 - \rho)$.

Clearly that limit exists if and only if $|\rho| < 1$, where ρ is called *radius of convergence of the power series*. The fourth identity can be proved instead by rewriting it as

$$\sum_{k=m}^{\infty} k \rho^k = \rho^m (m + \rho + \rho^2 + \cdots) \sum_{k=0}^{\infty} \rho^k = \rho^m \left(m - 1 + \sum_{k=0}^{\infty} \rho^k \right) \sum_{k=0}^{\infty} \rho^k,$$

and by invoking also the previous identity. Finally the last relation is the Maclaurin series of e^ρ , that is convergent not only for real ρ , but for all $\rho \in \mathbb{C}$. □