Steady-state quantities of interest of Markovian Queues

system type	M/M/1	M/M/m	M/M/∞	M/M/1/K
state space	$\mathbb{N}_0$	$\mathbb{N}_0$	$\mathbb{N}_0$	$\{0,1,\cdots K\} \subset \mathbb{N}_0$
mean arrival rate λ_i	$\lambda \ (\forall\ i)$	$\lambda \ (\forall\ i)$	$\lambda \ (\forall\ i)$	$\begin{cases} \lambda & (i < K) \\ 0 & (i \geqslant K) \end{cases}$
mean service rate μ_i	$\mu \ (\forall\ i)$	$\begin{cases} i\,\mu & (i < m) \\ m\mu & (i \geqslant m) \end{cases}$	$i\,\mu \ (\forall\ i)$	$\mu \ (\forall\ i)$
average service time $\overline{\vartheta}_s = \mathsf{E}[\vartheta_s]$	$\frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu}$	$\frac{1}{\mu}$
ergodicity conditions	$\varrho = \frac{\lambda}{\mu} < 1$	$\varrho = \frac{\lambda}{m\mu} < 1$	$\forall\ \varrho = \frac{\lambda}{\mu}$	$\forall\ \varrho = \frac{\lambda}{\mu}$
steady-state (s.s.) probabilities $\pi_i \equiv \Pr(x(\infty) = i)$	$(1-\varrho)\varrho^i \ (\forall\ i)$	$(*)$	$\frac{\varrho^i}{i!}e^{-\varrho} \ (\forall\ i)$	$\begin{cases} \varrho^i \frac{1-\varrho}{1-\varrho^{K+1}} & (i \leqslant K) \\ 0 & (i > K) \end{cases}$
idle probability $\pi_0 = \Pr(x(\infty) = 0)$	$(1-\varrho)$	$(*)$	$e^{-\varrho}$	$\frac{1-\varrho}{1-\varrho^{K+1}}$
utilization factor (prop. of time the sys. is busy) $\hat{U} = \Pr(x(\infty) > 0) = 1 - \pi_0$	ϱ	$1 - \pi_0$	$1 - e^{-\varrho}$	$\varrho\frac{1-\varrho^K}{1-\varrho^{K+1}}$
single server utilization $\tilde{\varrho} = \overline{x}_s/m$	ϱ	ϱ	0	$\varrho\frac{1-\varrho^K}{1-\varrho^{K+1}}$
blocking probability (prob. a costumer is rejected)	0	0	0	$\pi_K = \varrho^K \cdot \pi_0$
arrival rate within the resource $\lambda_{\text{in}} = \sum_{i=0}^{\infty} \lambda_i \pi_i$	λ	λ	λ	$\lambda - \lambda_{ab} = \lambda\frac{1-\varrho^K}{1-\varrho^{K+1}}$
abandonment rate $\lambda_{\text{ab}} = \lambda - \lambda_{\text{in}}$	0	0	0	$\lambda \cdot \pi_K = \lambda\frac{\varrho^K(1-\varrho)}{1-\varrho^{K+1}}$
throughput (productivity) $\eta = \hat{U} \cdot \mu \equiv \lambda_{\text{in}}$	λ	λ	λ	$\lambda\frac{1-\varrho^K}{1-\varrho^{K+1}}$
expected number of costumer in the system at s.s. $\overline{x} = \sum_{i=0}^{\infty} i \cdot \pi_i \equiv \lambda_{in} \cdot \overline{\vartheta}$	$\frac{\varrho}{1-\varrho}$	$m\varrho + \frac{m^m \varrho^{m+1}}{m!(1-\varrho)^2}\pi_0$	ϱ	$\frac{\varrho(1-(K+1)\varrho^K + K\varrho^{K+1})}{(1-\varrho^{K+1})\ (1-\varrho)}$
expected number of of busy servers at s.s. $\overline{x}_s = \lambda_{in} \cdot \overline{\vartheta}_s$	ϱ	$m\varrho$	ϱ	$\varrho\frac{1-\varrho^K}{1-\varrho^{K+1}}$
expected number of costumers in the buffer at s.s. $\overline{x}_b = \overline{x} - \overline{x}_b \equiv \lambda_{\text{in}} \cdot \overline{\vartheta}_b$	$\frac{\varrho^2}{1-\varrho}$	$\overline{x} - m\varrho$	0	$\frac{\varrho^2(1-K\varrho^{K-1} + (K-1)\varrho^K)}{(1-\varrho^{K+1})\ (1-\varrho)}$
expected time spent in the system $\overline{\vartheta} = \overline{x}/\lambda_{\text{in}}$	$\frac{1}{\mu\ (1-\varrho)}$	$\frac{\overline{x}}{\lambda}$	$\frac{1}{\mu}$	$\frac{(1-(K+1)\varrho^K + K\varrho^{K+1})}{\mu\ (1-\varrho^K)\ (1-\varrho)}$
expected time spent in the buffer $\overline{\vartheta}_b = \overline{\vartheta} - \overline{\vartheta}_s \equiv \overline{x}/\lambda_{\text{in}}$	$\frac{\varrho}{\mu\ (1-\varrho)}$	$\frac{\overline{x}}{\lambda} - \frac{1}{\mu}$	0	$\frac{\varrho(1-K\varrho^{K-1} + (K-1)\varrho^K)}{\mu\ (1-\varrho^K)\ (1-\varrho)}$

(*) Stationary state probabilities of a M/M/m queue system:

$$\pi_{s,0} = \frac{1}{\left(\sum_{i=0}^{m-1} \frac{m^i \varrho^i}{i!}\right) + \frac{m^m}{m!}\sum_{i=m}^{\infty} \varrho^i} = \frac{1}{\left(\sum_{i=0}^{m-1} \frac{m^i \varrho^i}{i!}\right) + \frac{m^m}{m!}\frac{\varrho^m}{(1-\varrho)}} \quad , \quad \begin{cases} \pi_{s,i} = \frac{m^i \varrho^i}{i!}\pi_{s,0} & (i < m) \\ \pi_{s,i} = \frac{m^m}{m!}\varrho^i \pi_{s,0} & (i \geqslant m) \end{cases}$$

Gordon-Newell Theorem: *Let X_i be some discrete random variable and $x_i \in \mathbb{Z}_{\geqslant 0}$ be a possible outcome for X_i. The probability distribution of costumers within an ergodic Gordon-Newell Network of v $M/M/m_i$ resources, at the steady-state, takes the following product form*

$$\pi_s(x_1, x_2, \ldots, x_v) = \Pr(\ X_1 = x_1, X_2 = x_2, \ldots, X_v = x_v\ ) = \kappa \cdot \prod_{i=1}^{v} \beta_i(x_i)$$

where κ is a normalization constant that is derived by the following constraint

$$\sum_{x_i \in N_{v,n}} \pi_s(x_1, x_2, \ldots, x_v) = \kappa \cdot \sum_{x_i \in N_{v,n}} \prod_{i=1}^{v} \beta_i(x_i) = 1 \quad \rightarrow \quad \kappa = \frac{1}{\sum_{x_i \in N_{v,n}} \prod_{i=1}^{v} \beta_i(x_i)}$$

where $N_{v,n}$ is the network sample space, whereas the functions $\beta_i(x_i)$ are selected in accordance with the next relations

$$\textit{if the i-th node is a M/M/1} \;:\; \beta_i(x_i) = \rho_i^{x_i} \quad \forall \quad x_i \in [0, n]$$

$$\textit{if the i-th node is a } M/M/m_i \;:\; \beta_i(x_i) = \begin{cases} \dfrac{\rho_i^{x_i}}{x_i!} & \text{if} \quad x_i < m_i \\ \dfrac{\rho_i^{x_i}}{m_i! m_i^{x_i - m_i}} & \text{if} \quad x_i \geqslant m_i \end{cases}$$

and $\rho_i = \lambda_i/\mu_i$ with λ_i such that $\boldsymbol{\lambda} = \boldsymbol{\lambda} \cdot \boldsymbol{R}$, where $\boldsymbol{R} \in \mathbb{R}^{v \times v}$ is routing probability matrix of the network and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_v) \in \mathbb{R}^v_{\geqslant 0}$. ∎

Markov's Inequality: *If X is a nonnegative random variable and $a > 0$, then the probability that X is at least a is at most the expectation of X, $\mu = \mathsf{E}[X]$, divided by a:*

$$\Pr(\mathrm{X} \geqslant \mathrm{a}) \leqslant \frac{\mu}{\mathrm{a}}$$

∎

Chebyshev's Inequality: *Let X (integrable) be a random variable with finite expected value μ and finite non-zero variance σ^2. Then for any real number $a > 0$,*

$$\Pr(|\mathrm{X} - \mu| \geqslant \mathrm{a}) \leqslant \frac{\sigma^2}{\mathrm{a}^2}.$$

∎

Kolgomorov's Inequality: *Let $X_1, \ldots, X_n$ be n i.i.d. r.v. with expected value $\mu = \mathsf{E}[X_i]$ and variance $\mathsf{Var}[X_i] < +\infty$ for $i = 1, \ldots, n$. Let $S_k = \sum_{i=1}^{k} X_i$ Then for any real number $\mathrm{a} > 0$,*

$$\Pr\left(\max_{1 \leqslant \mathrm{k} \leqslant \mathrm{n}} |\mathrm{S_k} - \mathsf{E}[\mathrm{S_k}]| \geqslant \mathrm{a}\right) \leqslant \frac{1}{\mathrm{a}^2} \cdot \mathsf{Var}[\mathrm{S_k}].$$

∎

68-95-99.7 rule: *In statistics, the 68-95-99.7 rule, also known as the empirical rule, is a shorthand used to remember the percentage of values that lie within a band around the mean in a normal distribution with a width of two, four and six standard deviations, respectively; more accurately, 68.27%, 95.45% and 99.73% of the values lie within one, two and three standard deviations of the mean, respectively.* ∎

Discrete distributions

distribution of $X = (S, p)$	Sample space S	pmf $p_X = \Pr(X = x_i)$	cdf $P_X = \Pr(X \leqslant x)$	mean $\mu_X = \mathsf{E}[X]$	variance $\sigma_X^2 = \mathsf{E}[(X - \mu_X)^2]$	MGF $M_X(z)$
non − par. r.v. X	$\{x_0, x_1, \ldots, x_n\}$	$\Pr(X = x_i)$	$\Pr(X \leqslant x)$	$\sum_{x_i \in X} x_i \cdot p_X(x_i)$	$\sum_{x_i \in X} (x_i - \mu_X)^2 \cdot p_X(x_i)$	$\sum_{k=0}^{n} p_X(x_k) \cdot z^{-k}$
$X \sim \mathrm{Ber}(q),\ q \in [0,1]$	$\{0, 1\}$	$\begin{cases} 1-q & X = 0 \\ q & X = 1 \end{cases}$	$\begin{cases} 0 & X < 0 \\ 1-q & X < 1 \\ 1 & X \geqslant 1 \end{cases}$	q	$q(1-q)$	$(1-q) + qz^{-1}$
$X \sim \mathrm{Geo}(q),\ q \in [0,1]$	$k \in \{0, 1, 2, \ldots\}$	$(1-q)^{k-1} q$	$1 - (1-q)^k$	$\frac{1}{q}$	$\frac{1-q}{q^2}$	$\frac{q}{z-(1-q)}$
$X \sim \mathrm{Bin}(n, q),\ q \in [0,1]$	$\{1, 2, \ldots, n\}$	$\binom{n}{k} (1-q)^{n-k} q^k$	$\sum_{i=0}^{\lfloor k \rfloor} q^i (1-q)^{n-i}$	nq	$nq(1-q)$	$((1-q) + qz^{-1})^n$
$X \sim \mathrm{Pois}(\lambda),\ \lambda \in \mathbb{R}_+$	$\{0, 1, 2, \ldots\}$	$\mathrm{e}^{-\lambda} \frac{\lambda^i}{k!}$	$\mathrm{e}^{-\lambda} \sum_{i=0}^{\lfloor k \rfloor} \frac{\lambda^i}{i!}$	λ	λ	–
$X \sim \mathrm{Uni}(a, b),\ a, b \in \mathbb{Z}, a \leqslant b$	$k \in \{a, a+1, \ldots, b-1, b\}$	$\frac{1}{b-a+1}$	$\frac{\lfloor k \rfloor - a + 1}{b-a+1}$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2+1}{12}$	$\frac{z^{-a} - z^{-b-1}}{(b-a+1)(1-z^{-1})}$

Continuous distributions

distribution of $X = (S, p)$	Sample space S	pdf $f_X = \mathrm{Pr}(\mathrm{X} = \mathrm{x_i})$	cdf $F_X = \mathrm{Pr}(\mathrm{X} \leqslant \mathrm{x})$	mean $\mu_X = \mathsf{E}[X]$	variance $\sigma_X^2 = \mathsf{E}[(X - \mu_X)^2]$	MGF $M_X(s)$
$X \sim \mathrm{Exp}(\lambda)$	$\mathbb{R}_{\geqslant 0}$	$1 - \mathrm{e}^{-\lambda x}$	$\lambda \cdot \mathrm{e}^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{s+\lambda}$
$X \sim \mathrm{Erl}(\mathrm{k}, \lambda),\ \mathrm{k} \in \mathbb{N}_{>0},\ \lambda \in \mathbb{R}_+$	$\mathbb{R}_{\geqslant 0}$	$\frac{\lambda^k}{(k-1)!} x^{k-1} \mathrm{e}^{-\lambda \mathrm{x}}$	$1 - \sum_{n=0}^{k-1} \frac{1}{n!} \mathrm{e}^{-\lambda \mathrm{x}} (\lambda \mathrm{x})^{\mathrm{n}}$	$\frac{k}{\lambda}$	$\frac{k}{\lambda^2}$	$\frac{\lambda^k}{(s+\lambda)^k}$
$X \sim \mathrm{Norm}(\mu, \sigma^2),\ \mu_{\mathrm{X}} \in \mathbb{R},\ \sigma \in \mathbb{R}_{>0}$	$\mathbb{R}$	$\frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{(\mathrm{x}-\mu)^2}{2\sigma^2}}$	$\frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\mathrm{x}-\mu}{\sigma\sqrt{2}}\right)\right]$	μ	σ^2	–
$X \sim \mathrm{Uni}(\mathrm{a}, \mathrm{b}),\ \mathrm{a}, \mathrm{b} \in \mathbb{Z}, \mathrm{a} \leqslant \mathrm{b}$	$[a, b]$	$\frac{1}{b-a}$	$\frac{x-a}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	–

Probability formulas

Probability definition	$\mathrm{Pr}(\mathrm{A}) = \lim_{\mathrm{N}\to\infty} \frac{\mathrm{N_A}}{\mathrm{N}}$
Addition rule	$\mathrm{Pr}(\mathrm{A} \cup \mathrm{B}) = \mathrm{Pr}(\mathrm{A}) + \mathrm{Pr}(\mathrm{B}) - \mathrm{Pr}(\mathrm{A} \cap \mathrm{B})$
Conditional probability	$\mathrm{Pr}(\mathrm{A} \mid \mathrm{B}) = \frac{\mathrm{Pr}(\mathrm{A} \cap \mathrm{B})}{\mathrm{Pr}(\mathrm{B})}, \quad \mathrm{Pr}(\mathrm{B}) > 0$
Law of total probability	$\mathrm{Pr}(\mathrm{A}) = \sum_{\forall\, \mathrm{i}} \mathrm{Pr}(\mathrm{B_i}) \cdot \mathrm{Pr}(\mathrm{A} \mid \mathrm{B_i})$
Bayes' Theorem	$\mathrm{Pr}(\mathrm{B_j} \mid \mathrm{A}) = \frac{\mathrm{Pr}(\mathrm{A} \cap \mathrm{B_j})}{\sum_{\forall\, \mathrm{i}} \mathrm{Pr}(\mathrm{B_i}) \cdot \mathrm{Pr}(\mathrm{A} \mid \mathrm{B_i})}$

List of mathematical series

Proposition: *Let k, m, $n \in \mathbb{N}$ and $\rho \in \mathbb{R}$ the following identities holds:*

- *Arithmetic progression:*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

- *Finite geometric progression:*

$$\sum_{k=m}^{n} \rho^k = \begin{cases} \frac{1-\rho^{n-m-1}}{1-\rho}\rho^m & \text{if } \rho \neq 1 \\ n-m+1 & \text{if } \rho = 1 \end{cases}$$

- *Infinite geometric progression:*

$$\sum_{k=m}^{\infty} \rho^k = \frac{1}{1-\rho}\rho^m \quad \text{if } |\rho| < 1$$

- *Weighted infinite geometric progression:*

$$\sum_{k=m}^{\infty} k\rho^k = \frac{m+\rho-m\rho}{(1-\rho)^2}\rho^m \quad \text{if } |\rho| < 1$$

- *Exponential series:*

$$\sum_{k=0}^{\infty} \frac{\rho^k}{k!} = \mathrm{e}^{\rho}$$

■

Proof: The first identity is an elementary result of math. Legend says that it was proved also by the young J.C.F Gauss. Its proof is as follows. Let us first note that the progression

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n$$

can be rewritten as $S_n = a_n + a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1$.
Thus, one has that because of $(a_1 + a_n) = (a_i + a_{n-i+1})$ for all $i = 1, 2, \ldots, n$, then $2S_n = n(a_1 + a_n)$. By letting $a_1 = 1$, and $a_n = n$ one derives the first identity.
The second identity can proved by noticing that $(1-\rho^{n+1}) = (1-\rho)(1+\rho+\rho^2+\cdots+\rho^n)$, while taking into account that

$$\sum_{k=m}^{n} \rho^k = \rho^m \sum_{k=0}^{n-m} \rho^k.$$

The third identity follows by the evaluating the limit of the second relation as $n \to \infty$. Let us further note that, in this case, if $m = 0$, one get that $\sum_{k=0}^{\infty} \rho^k = 1/(1-\rho)$.
Clearly that limit exists if and only if $|\rho| < 1$, where ρ is called *radius of convergence of the power series.* The fourth identity can be proved instead by rewriting it as

$$\sum_{k=m}^{\infty} k\rho^k = \rho^m(m+\rho+\rho^2+\cdots)\sum_{k=0}^{\infty} \rho^k = \rho^m \left(m - 1 + \sum_{k=0}^{\infty} \rho^k\right) \sum_{k=0}^{\infty} \rho^k,$$

and by invoking also the previous identity. Finally the last relation is the Maclaurin series of e^{ρ}, that is convergent not only for real ρ, but for all $\rho \in \mathbb{C}$. □