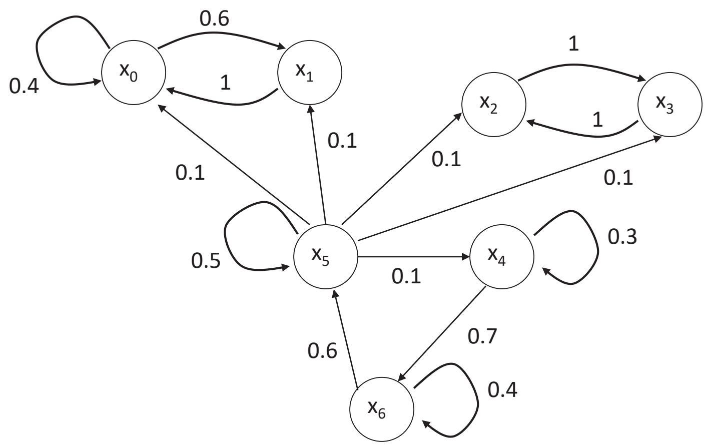
Stochastic Models

A.A. 2019/2020

Assignment 4

Exercise 1. Consider the Discrete-Time Markov Chain (DT-MC) depicted in the next figure.



(a) Evaluate the strongly connected components. Which are ergodics and which transients?

(b) Evaluate the periodicity of the ergodic components

(c) Evaluate if the given DT-MC is ergodic and explain what this property implies

(d) Evaluate if, by the change of the weights of some of the existing arc, the given analysis would change

(e) In the case that the reply to previous question is negative, evaluates if it is possible make the given DT-MC ergodic by simply changing the orientation of a single arc.

Exercise 2. Marital status for a man in his 30s is considered using the following Markov process $\{X_\tau, \tau = 30, 31, \ldots\}$. The state space of the process consists of three states: Single ($X_\tau = x_0$), Cohabiting ($X_\tau = x_1$), Married ($X_\tau = x_2$). The transition matrix from year to year has the following form:

$$\boldsymbol{P} = \begin{pmatrix} 0.6 & 0.35 & 0.05 \\ 0.20 & 0.70 & 0.10 \\ 0.10 & 0.05 & 0.85 \end{pmatrix}$$

Draw the transition graph for this Markov chain and then answer the following questions:

(a) Calculate the probability that a 35-year-old married man gets divorced next year, and marries again the following year.

(b) Given a 30-year-old man is single this year, calculate the the probability that he will still be single in two years time.

(c) Do you think this is a good model for the marital status of a man? What is good/bad about it?

Exercise 3. In a factory, a processing system consisting of a single server may work only one piece at a time. Every 5 seconds a single piece may arrive to the system with probability $a = 0.7$. With probability $b = 0.6$ the machine performs correctly its operation, then the piece goes out the system. Otherwise, with probability $1 - b$ the processing task is repeated and then the piece goes out the system.

It may occur that at the same time one piece leaves the system and another arrives.

(a) Provides a graphical representation of the DT-MC associated with this stochastic process and determines the transition probability matrix.

(b) Is this process ergodic? If yes, evaluates its limiting distribution. If not evaluate its stationary distribution.

Exercise 4. A manufacturing system consists of a single production machine. At each time T, the arrival of a new piece is modelled as a Bernoulli trial with parameter $p = 0.8$.

The machine has two identical servers. If one server at time T is busy, it will complete its task within the next time interval with probability $q = 0.9$. The machine has no waiting lines, thus if the two servers are both busy a new arrival is discharged.

(a) Model this stochastic process as a DT-MC. Drawn its transition graph and calculate its transition probability matrix (express probabilities as functions of p and q).

(b) Is the DT-MC ergodic? Specify what the ergodicity property implies in practice.

(c) Calculate the stationary probability distribution.

(d) Calculate the probability that an incoming piece has to be discarded when the machine has reached its steady state probability regime.

Exercise 5. Let $X = \{x_1, x_2, x_3\}$ be the vertex set associated to a given DT-MC which transition probability matrix is:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.9 & 0 \end{bmatrix}.$$

Calculate:

(a) the edge set $E \subset X \times X$ of the graph associated with the given DT-MC.

(b) if the given DT-MC is ergodic, or not, through both the grafical and analitic criteria.

(c) the stationary distribution of the given markov chain.

Exercise 6. Consider a service station consisting of a robot arm and a pre-load buffer with capacity of 1 piece. A transfer line, at each time cycle $\tau = 10$ seconds, introduces pieces in the buffer with probability p. The piece is introduced into the station's buffer if only if the robot is free, otherwise the piece is discharged. The robot is able to complete its task in a cycle time. This operation may fail with probability q. If thus, it is needed another cycle time to repeat the task. This operation may fail again with the same probability.

(a) Model the buffer state as a DT-MC, drawn its transition graph and calculate its transition probability matrix

(b) Let $p = 0.9$ e $q = 0.2$, evaluate if this DT-MC is ergodic by means of the eigenvalue criteria.

(c) Calculate the marginal stationary distribution of this stochastic process

(d) Let η_d be the percentage of pieces discharged in steady-state with respect to the number of incoming pieces. Calculate η_d.

(e) Let $\bar{x}$ be the mean number of pieces within the buffer in steady condition. Calculate $\bar{x}$.

(f) Assume that the robot may fail its task at most once. By taking into account this aspect, model this new process as a DT-MC.

Exercise 7. In a communication system a station T_t transmits every Δ seconds messages with fixed length to a receiving station T_r through a noisy communication channel. The transmission time is negligible. The probability that a message sent by T_t is received by T_r is $p = 0.8$. Station T_r is able to process correctly the received message in a time cycle with probability $q = 0.6$. Otherwise another period of Δ [sec] is needed to correctly terminate the elaboration. The capacity of the T_r station equal to 1 message.

(a) Assume the stochastic process describing the state of station T_r be represented by the following three states: 0 : No messages to be elaborated; 1_E : Elaborating a message; 1_R : Re-elaborating a message. Model the $\{X_\tau, \tau = 0, \Delta, 2\Delta, \ldots\}$ as a DT-MC, draw its transition graph and calculate its transition probability matrix.

(b) Is this DT-MC ergodic? If yes, calulates its limiting probability distribution.

(c) Calculate the average number of messages in the T_r station at the steady state.

(d) Calculate the rate of correctly elaborated messages at the steady state.

(e) Under the assumption that also the second elaboration could be modelled as a Bernoulli trial, modify the transition graph of the DT-MC to take into account this aspect. Is this DT-MC ergodic?

(f) Calculate the rate of correctly elaborated messages at the steady state, then compare and discuss this result with that obtained at point (d).

(g) Assume that the T_r station has buffer capacity of 2 messages. Assume that: i) only one message can be elaborated at time; ii) the second elaboration always terminates correctly. Draw the transition graph for this newer scenario. Hint: the DT-MC would have 5 states.

Solutions

Solution of Exercise 1. Remanding the following defintions:

A directed graphs is said to be strongly connected if every vertex is reachable (i.e., exists a sequence of arcs) from every other vertex.

The Strongly Connected Components (SCCs)of a directed graph forms a partition into subgraphs that are themselves strongly connected.

It follows that the SCCs associated with the given DT-MC are, resp.,

$$\begin{aligned} S_1 &= \{x_0, x_1\} \\ S_2 &= \{x_2, x_3\} \\ S_3 &= \{x_4, x_5, x_6\} \end{aligned}$$

Moreover, since a SCC is said to be "transitory" if, once leaved, it is not possible reach it again by following any path. Whereas it is "ergodic" or "absorbing" if, once reached, there are no paths enabling the possibility to leave it, then S_1 and S_2 are ergodic, whereas S_3 is transient.

Since, S_1 has at least a self it follows that it is aperiodic. On the other hand S_2 is periodic with period $d = 2$.

The given DT-MC is not erogodic because it is reducible, namely it has more then one ergodic SCCs. The ergodicity is a property of the MC which implies that its limiting distribution is unique, and does not depend on the particular initial distribution $\pi(0) = [\pi_0, \pi_1, \pi_2, \ldots]$. Thus it results that to understand the behaviour of an ergodic MC chain, we can simply study one their realizations. However, this is not our case.

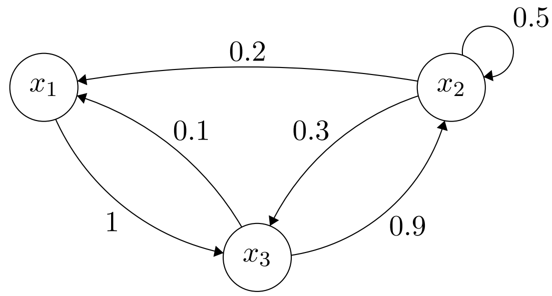
Let us now note that, there is no possibility to make the given DT-MC ergodic by the change of the weight of an existing arc, because this property does not depend by the MC's probability.

On the other hand, if we change the orientation of either $x_5 \to x_2$ to $x_2 \to x_5$ or $x_5 \to x_3$ to $x_3 \to x_5$ then, the resulting DT-MC becomes ergodic because it now has a single ergodic, aperiodic component denoted by $S_1 = \{x_0, x_1\}$. Note that the change of the orientation of either $x_5 \to x_0$ or $x_5 \to x_1$ do not make the DT-MC ergodic because S_2 is periodic, and thus its stationary distribution would depend on the given initial marginal distribution.

Solution of Exercise 2. Let p_{ij} be the (i, j) entry of the transition probability matrix of the given DT-MC, the edge set of its associated graph consists of all the edges (i, j) such that $p_{ij} > 0$, then it results that

$$E = \{(x_1, x_3), (x_2, x_1), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_3, x_1)\} \subseteq X \times X.$$

The resulting graph $G(X, E)$ is provided below:



Since every vertex of the given graph is reachable from every other vertex, then the graph is strongly connected, moreover, since the graph is aperiodic, then the DT-MC is ergodic, or equivalently irreducible.

This aspect is confirmed by the eigenvalues' criteria, in fact, by solving

$$\det\left\{\begin{bmatrix} s & 0 & -1 \\ -0.2 & s+0.5 & -0.3 \\ -0.1 & -0.9 & s \end{bmatrix}\right\} = s^3 - s^2/2 - (37s)/100 - 13/100 = 0$$

it results that the eigenvalues of $\boldsymbol{P}$ are, resp.,

$$\lambda_1 = 1 \quad , \quad \lambda_{2,3} = \frac{-0.5 \pm \sqrt{-0.27}}{2}$$

Since $|\lambda 2, 3| < 1$ then, also the analytic criteria confirms the ergodicity of the given DT-MC.

Finally, let $\mathbf{1}$ be a column vector of 1 with proper dimension, to calculate the stationary distribution $\pi_s = (\pi_{s,1}, \pi_{s,2}, \pi_{s,3})$ of the given DT-MC, we have to solve the following system of equations

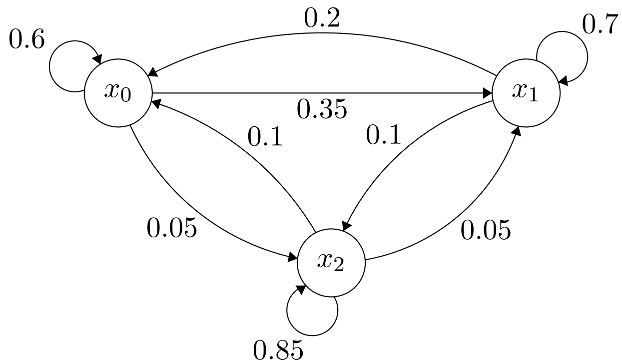
$$\left\{\begin{array}{l} \pi_s = \pi_s \cdot \boldsymbol{P} \\ \pi_s \cdot \mathbf{1} = 1 \end{array}\right. \rightarrow \left\{\begin{array}{l} \pi_{s,1} = 0.2\pi_{s,2} + 0.1\pi_{s,3} \\ \pi_{s,2} = 0.5\pi_{s,2} + 0.9\pi_{s,3} \\ \pi_{s,3} = 0.1\pi_{s,1} + 0.3\pi_{s,2} \\ \pi_{s,1} + \pi_{s,2} + \pi_{s,3} = 1 \end{array}\right. \rightarrow \pi_s = \left(\frac{23}{163}, \frac{90}{163}, \frac{50}{163}\right). \tag{1}$$

Notice that, since the DT-MC is ergodic its limiting distribution $\pi_l \equiv \pi_s$.

Solution of Exercise 3. The transition graph associated with this DT-MC is as follows

The probability that a man which status is "married" at 35 gets divorsed at 36 and then remarries at 37 years old, is equivalent to the following conditional probability

$$\Pr(X_{37} = x_2, X_{36} \neq x_2 | X_{35} = x_2).$$



By inspecting the above transition graph we have that

$$\begin{aligned}\Pr(X_{37} = x_2, X_{36} \neq x_2 | X_{35} = x_2) &= \\ &= \Pr(X_{37} = x_2 | X_{36} = x_0)\Pr(X_{36} = x_0 | X_{35} = x_2) \\ &\quad +\Pr(X_{37} = x_2 | X_{36} = x_1)\Pr(X_{36} = x_1 | X_{35} = x_2) \\ &= p_{20} \cdot p_{02} + p_{21} \cdot p_{12} = 0.1 \cdot 0.05 + 0.05 \cdot 0.1 = 0.01\end{aligned}$$

Notice that this probability, differs to the probability that a man which status was "married" at 35 will remain married with same person at 37, that is instead

$$\begin{aligned}\Pr(X_{37} = x_2, X_{36} = x_2 | X_{35} = x_0) &= \Pr(X_{37} = x_2 | X_{36} = x_2)\Pr(X_{36} = x_2 | X_{35} = x_2) \\ &= p_{22} \cdot p_{22} = 0.85 \cdot 0.85 = 0.7225\end{aligned}$$

Moreover, notice that the above two probabilities differs also to the probability that a man which status is "married" at 35 will be married at 37 that is instead the "2-step return probability" to state x_2 that is

$$p_{22}^{(2)} = \Pr(X_{37} = x_2 \mid X_{35} = x_2) = 0.01 + 0.7225 = 0.7325$$

The probability that a single 30-year old man will still be single in two years, by the total probability law is :

$$\begin{aligned}p_{00}^{(2)} = \Pr(X_{32} = x_0 | X_{30} = x_0) &= \Pr(X_{32} = x_0 | X_{31} = x_0)\Pr(X_{31} = x_0 | X_{30} = x_0) + \\ &\quad +\Pr(X_{32} = x_0 | X_{31} = x_1)\Pr(X_{31} = x_1 | X_{30} = x_0) + \\ &\quad +\Pr(X_{32} = x_0 | X_{31} = x_2)\Pr(X_{31} = x_2 | X_{30} = x_0) \\ &= 0.6 \cdot 0.6 + 0.35 \cdot 0.2 + 0.05 \cdot 0.1 = 0.4350\end{aligned}$$

This quantity could be easily evaluated by means of the N-steps Transition Probability, thus by exploiting the Chapman Kolmogorov Relation as follows

$$\boldsymbol{P}^2 = \begin{pmatrix} \Pr(X_{m+2} = x_0 | X_m = x_0) & \Pr(X_{m+2} = x_1 | X_m = x_0) & \Pr(X_{m+2} = x_2 | X_m = x_0) \\ \Pr(X_{m+2} = x_0 | X_m = x_1) & \Pr(X_{m+2} = x_1 | X_m = x_1) & \Pr(X_{m+2} = x_2 | X_m = x_1) \\ \Pr(X_{m+2} = x_0 | X_m = x_2) & \Pr(X_{m+2} = x_1 | X_m = x_2) & \Pr(X_{m+2} = x_2 | X_m = x_2) \end{pmatrix} =$$

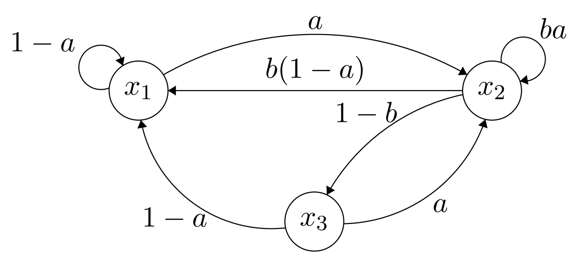
$$= \begin{pmatrix} 0.4350 & 0.4575 & 0.1075 \\ 0.2700 & 0.5650 & 0.1650 \\ 0.1550 & 0.1125 & 0.7325 \end{pmatrix}$$

Finally, let us note that the assumption of Markovianity for this stocastic process is dubious. For instance a man who has cohabited with his partner for many years is more likely to get married than a man who has cohabitted for only one year.

Solution of Exercise 4. This stochastic process $\{X_\tau, \tau = 0, 5, 10, 15, \ldots\}$ could be modelled by a DT-MC which state space consits of three states, resp.,

- Idle-mode (x_1): no pieces in the system
- Processing-mode (x_2): one piece in the works
- Re-processing-mode (x_3): because of the previous work sequence has failed, that piece is still in the works.

The transition graph for the considered DT-MC is provided below



Notice that, a is the probability that a piece arrives, and b is the probability that the works goes well. Because of an arrival or a departure of a piece can occur simultaneously, and they are independent events, then

$$p_{22} = \Pr(X_\tau = x_2 | X_{\tau-1} = x_2) = b \cdot a = 0.6 \cdot 0.7 = 0.42$$

On the other hand, and because of they are independent events, the probability that a piece is correctly processed but there is not a new piece to be processed is

$$p_{21} = \Pr(X_\tau = x_1 | X_{\tau-1} = x_2) = b(1-a) = 0.6 \cdot 0.3 = 0.18$$

.

Thus it further results that

$$p_{23} = \Pr(X_\tau = x_3 | X_{\tau-1} = x_2) = 1 - p_{22} - p_{21} = 1 - b = 0.4$$

.

The transition probability matrix for this process is thus as follows

$$\boldsymbol{P} = \begin{pmatrix} 0.3 & 0.7 & 0 \\ 0.18 & 0.42 & 0.4 \\ 0.3 & 0.7 & 0 \end{pmatrix}$$

Let us now focus our attention on the drawn transition graph. Since every vertex is reachable from every other vertex, and there is at least a self-loop then this DT-MC is not only irriducible because of the graph is strongly connected, but also it is aperiodic. Then this DT-MC is ergodic.

It follows that for this DT-MC its limiting distribution Π_ℓ does not depend by its initial probability distribution $\Pi(0)$, thus it coincides with its stationary probability distribution Π_s, namely $\Pi_\ell = \Pi(\tau)|_{\tau\to\infty} \equiv \Pi_s$.

Thus, the limiting distribution can be evaluated by solving the following linear system

$$\begin{cases} \Pi_s = \Pi_s \cdot \boldsymbol{P} \\ \Pi_s \cdot \boldsymbol{1} = 1 \end{cases} \to \begin{cases} \pi_{s,1} = 0.3\pi_{s,1} + 0.18\pi_{s,2} + 0.3\pi_{s,3} \\ \pi_{s,2} = 0.7\pi_{s,1} + 0.42\pi_{s,2} + 0.7\pi_{s,3} \\ \pi_{s,3} = 0.4\pi_{s,2} \\ \pi_{s,1} + \pi_{s,2} + \pi_{s,3} = 1 \end{cases} \tag{2}$$

By solving the above linear system by the substitution method it results that

$$\begin{cases} 0.7\pi_{s,1} = 0.18\pi_{s,2} + 0.3\pi_{s,3} \\ 0.58\pi_{s,2} = 0.7(\pi_{s,1} + \pi_{s,3}) \\ \pi_{s,3} = 0.4\pi_{s,2} \\ \pi_{s,1} + \pi_{s,2} + \pi_{s,3} = 1 \end{cases} \to \begin{cases} \pi_{s,2} = \frac{0.7}{0.58}(\pi_{s,1} + \pi_{s,3}) \\ \pi_{s,3} = 0.4\pi_{s,2} \\ \pi_{s,1} + \pi_{s,2} + \pi_{s,3} = 1 \end{cases} \to \tag{3}$$

$$\begin{cases} \pi_{s,2} = \frac{0.7}{0.58}(\pi_{s,1} + 0.4\pi_{s,2}) \\ \pi_{s,3} = 0.4\pi_{s,2} \\ \pi_{s,1} + \pi_{s,2} + \pi_{s,3} = 1 \end{cases} \to \begin{cases} (1 - \frac{0.4\cdot 0.7}{0.58})\pi_{s,2} = \frac{0.7}{0.58}\pi_{s,1} \\ \pi_{s,3} = 0.4\pi_{s,2} \\ \pi_{s,1} + \pi_{s,2} + \pi_{s,3} = 1 \end{cases} \to \tag{4}$$

$$\begin{cases} 0.5172 \cdot \pi_{s,2} = 1.207 \cdot \pi_{s,1} \\ \pi_{s,3} = 0.4\pi_{s,2} \\ \pi_{s,1}\left(1 + \frac{1.207}{0.5172} + 0.4 \cdot \frac{1.207}{0.5172}\right) = 1 \end{cases} \to \begin{cases} \pi_{s,2} = \frac{1.207\cdot 0.2337}{0.5172} \approx 0.55 \\ \pi_{s,3} = 0.4 \cdot 0.55 = 0.22 \\ \pi_{s,1} = 0.23 \end{cases} \tag{5}$$

Thus it results that

$$\Pi_\ell = \Pi_s \approx (0.23, 0.55, 0.22)\,.$$

Solution of Exercise 5. Since the manufacturing process can operate at most two pieces at time and there is no waiting line, the stochastic process $\{X_\tau, \tau = T, 2T, \ldots\}$ describing the functioning of this process can be modelled by a three state DT-MC which takes into account the number of piece in the process, namely, $X_\tau \in X = \{0, 1, 2\}$.

Because of both a new arrival and the service process are modelled as a Bernoulli trials with parameters p and q, and because of independence, the probability that the piece under works does not terminate its process while a new piece is arrived corresponds to

$$p_{12} = \Pr(X_\tau = 2 | X_{\tau-1} = 1) = p \cdot (1 - q).$$

On the other hand we have that

$$p_{10} = \Pr(X_\tau = 0 | X_{\tau-1} = 1) = q \cdot (1 - p)$$

which corresponds to the probability that there were only a piece in the process at time τ, its processing is terminated correctly while no further pieces are arrived.

Finally, by exclusion it results that

$$p_{11} = \Pr(X_\tau = x_1 | X_{\tau-1} = x_1) = 1 - p(1 - q) - q(1 - p) = pq + (1 - q)(1 - p),$$

which correspond to the probability that the current process is terminated correctly a new piece is arrived, or alternatively, the process is not terminated within T and no further pieces are arrived.

Similarly, it results that

$$p_{20} = \Pr(X_\tau = 0 | X_{\tau-1} = 2) = q^2(1 - p)$$

which corresponds to the probability that there were two pieces in the process at time τ, both the processes are terminated correctly while a no further pieces are arrived.

On the other hand we have that

$$p_{21} = \Pr(X_\tau = 1 | X_{\tau-1} = 2) = 2 \cdot (q \cdot (1 - q)(1 - p) \cdot q^2) + q^2 \cdot p$$

which corresponds to the probability that:

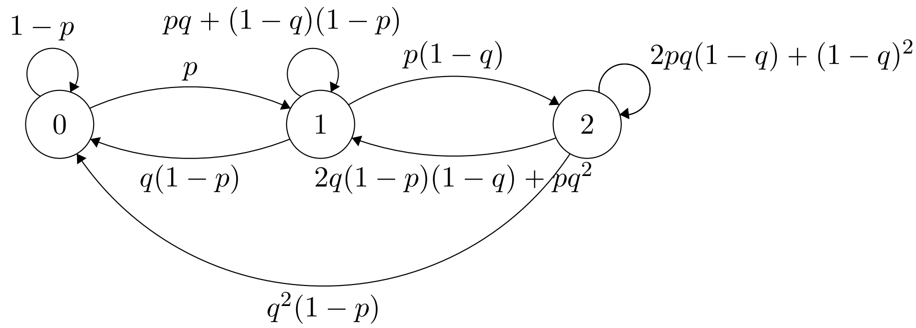
- there were two pieces in the process at time τ, only one process terminates correctly, the other not, and a no further pieces are arrived. This event is counted twice because it may correspond on either the first machine terminates correctly and the other not, or vice-versa
- there were two pieces in the process at time τ, both the processes terminates correctly within τ and a piece is arrived.

Similarly, it results that

$$p_{22} = \Pr(X_\tau = 2 | X_{\tau-1} = 2) = p(1-q)^2 + (1-p)(1-q)^2 + 2 \cdot pq(1-q) = (1-q)^2 + 2 \cdot pq(1-q),$$

which corresponds to the probability that both the processes do not terminate correctly independently on the fact that a new piece is arrived, or one of the two process is terminated correctly and a new piece is arrived, counted twice.

The transition graph associated with this DT-MC is provided below:



It can be noted that the resulting graph is strongly connected because each state is reachable from another by an oriented path. It results that the DT-MC is irriducible. Moreover, because there are self-loops we can immediatly assert that the strongly connected component associated with this DT-MC is also aperiodic, and thus the DT-MC is ergodic. It follows that, for this process, we can estimate the long term marginal unconditional probability distributions, and other performance metrics, independently to the system's initial probability distribution.

Let us now evaluate the corresponding limiting probability distribution Π_ℓ. Let us first define the transition matrix of the DT-MC as follows

$$\boldsymbol{P} = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.18 & 0.74 & 0.08 \\ 0.162 & 0.684 & 0.154 \end{pmatrix}$$

Then, since the DT-MC is ergodic then Π_ℓ correspond to its stationary distribution Π_s. Thus, by solving the following linear system

$$\begin{cases} \Pi_s = \Pi_s \cdot \boldsymbol{P} \\ \Pi_s \cdot \mathbf{1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,0} = 0.2\pi_{s,0} + 0.18\pi_{s,1} + 0.162\pi_{s,2} \\ \pi_{s,1} = 0.8\pi_{s,0} + 0.74\pi_{s,1} + 0.684\pi_{s,2} \\ \pi_{s,2} = 0.08\pi_{s,1} + 0.154\pi_{s,2} \\ \pi_{s,0} + \pi_{s,1} + \pi_{s,2} = 1 \end{cases} \quad (6)$$

it results that

$$\Pi_\ell \equiv \Pi_s \approx (0.1824, 0.7470, 0.0706)\,.$$

Let us now calculate the probability that an incoming piece has to be discarded when the machine has reached its steady state probability distribution Π_ℓ. Well, for sure this event will implies that my DT-MC will remain in state 2, however this quantity differs from p_{22} because of it counts also the possibility that a new piece enters in the system because of one of the two servers terminates its process. Thus, it follows that

$$\Pr(1 \text{ piece is discharged}) = (1-q)^2 \cdot \pi_2(\infty) = (1-q)^2 \cdot \pi_{2,\ell} = 0.1^2 \cdot 0.0706 = 0.0706.$$

Let us further note that the above probability differs to the probability that a piece would arrive and then it is discharged, that is instead

$$\Pr(\text{a piece would arrive and then it is discharge}) = p(1-q)^2\pi_2(\infty) = 0.0706 \cdot 0.8 = 0.05648$$

Finally note that, most of the previous analysis can be done on MatLab. Here a list of instruction that can help the analysis.

```
clc, clear all, close all
% Define the transtion probability matrix
P=[0.2 0.8 0;0.18 0.74 0.08; 0.162 0.684 0.154]
% Cardinality of the sample space of the DT-MC
n=length(P)
% Evaluate the eigenvalues of P
eig(P)
% Simple algorithm to evaluate if the DT-MC is ergodic
sum( abs(eig(P))>1 )>1
% To evaluate the stationary distribution Pi_s numerically we can simply
% use the Chapman-Kolmogorov equation. In particular, let k_end be a
% suffienctly big number of iteration
k_end=100;
% and let the initial probability distribution of DT-MC be, for instance,
Pi0=1/n*ones(1,n)
% then
Pi_s=Pi0*P^k_end

% Moreover its also possible simulate the behaviour of a DT-MC as follow:
k_end=10;                   % Simulation lenght
pi_0=[1 0 0];               % Initial marginal probability
vec_pi=zeros(k_end+1,n);    % DT-MC evolution vector
vec_pi(1,:)=pi_0;           % Initialization of vec_pi @ k=0
for k=2:1:k_end+1
    vec_pi(k,:)=vec_pi(k-1,:)*P % DT-MC state update
end
```

```
% Notice that
round(vec_pi(k_end,:),5)==round(pi_0*P^(k_end),5)

figure(1)
subplot(311)
stairs(0:k_end,vec_pi(:,1))
ylim([0 1])
ylabel('$\pi_1(k)$','fontsize',14,'interpreter','latex')
set(gca,'FontSize',18,'FontName','times')
subplot(312)
stairs(0:k_end,vec_pi(:,2))
ylim([0 1])
ylabel('$\pi_2(k)$','fontsize',14,'interpreter','latex')
set(gca,'FontSize',18,'FontName','times')
subplot(313)
stairs(0:k_end,vec_pi(:,3))
ylim([0 1])
xlabel('# Iteration')
ylabel('$\pi_3(k)$','fontsize',14,'interpreter','latex')
set(gca,'FontSize',18,'FontName','times')
```

Let us further note that on MatLab exists also a dedicated tool-box which helps the analysis of DT-MC. Hereinafter some of their commands are listed.

```
clc, clear all, close all
% Define the transtion probability matrix
P=[0.2 0.8 0;0.18 0.74 0.08; 0.162 0.684 0.154]
% Cardinality of the sample space of the DT-MC
n=length(P)
mc=dtmc(P)
% Evaluate if it is "erogodic" or not
tf = isergodic(mc)
% Evaluate if it is "reducible" or not
tf = isreducible(mc)
% Draw its transition graph, while classifying the number and type of
% trongly connected components. The edge probabiliteis color correspond
% to the right legend
figure(1)
graphplot(mc,'ColorNodes',true,'ColorEdges',true);
% Draw the eigenvalues of P within the unit circle
figure(2)
```

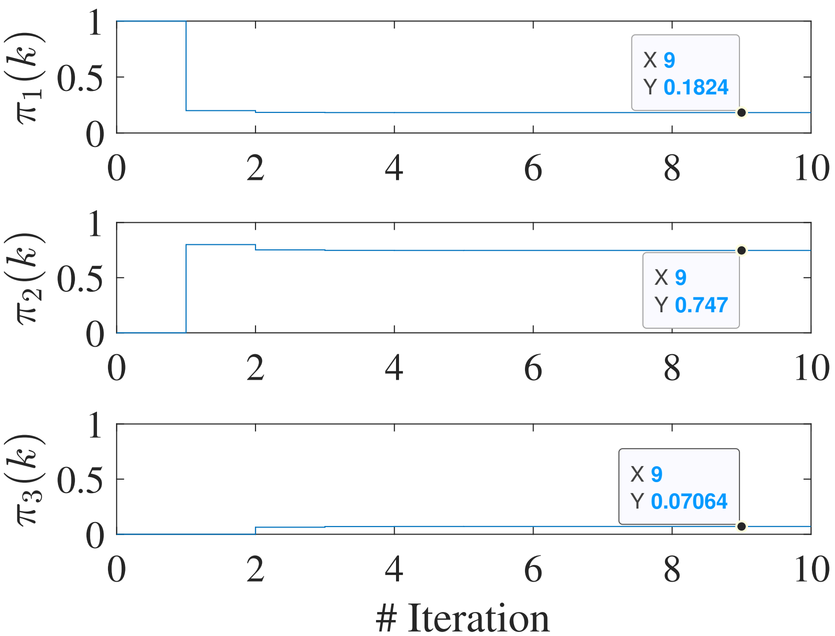


Figure 1: Output figure generated by the above MatLab script which show the temporal evolution of the marginal probabilies of the DT-MC. It is evident that, after only 10 iterations the system has reached its steady limiting distribution, namely $\Pi(\infty) \equiv \Pi_s = \Pi_\ell = (0.1824, 0.7470, 0.0706)$.

```
eigplot(mc)
% By means of the command "simulate" we can generate n random walk of lenght
k_end=20
% each staring by one of the n states.
X = simulate(mc,k_end,'X0',ones(1,n))
```

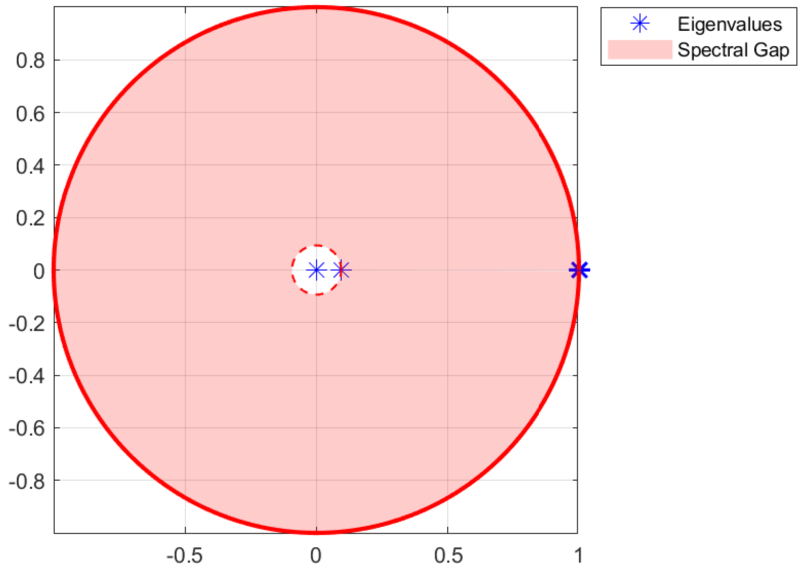


Figure 2: Output of the command "eigplot(mc)".

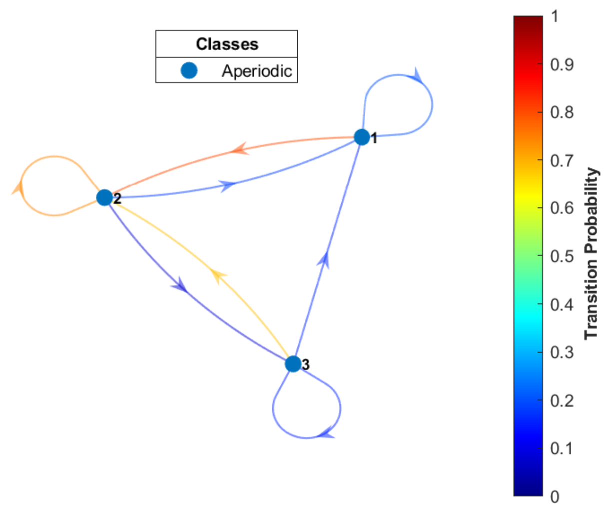
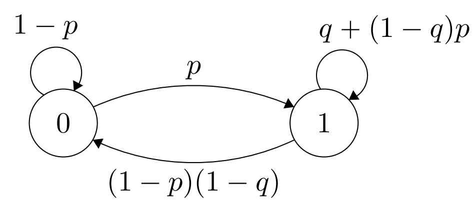


Figure 3: Output of the command "graphplot(mc,'ColorNodes',true,'ColorEdges',true)".

Solution of Exercise 6. Since an incoming piece can be introduced into the buffer if and only if the robot is free, and that no more than a piece may stay within the system. It follows that this stochastic process $\{X_\tau, \tau = T, 2T, \ldots\}$ associated with the buffer state may assume only two values, either 0, or 1, representing the number of pieces in the station. IT follows that $X = \{0, 1\}$. Let us further note that the event of having an incoming piece from the transfer line, that has probability equal to p, and the event that the robot fails its task, that has probability q, are clearly independent. Let us further note events are scheduled at discrete-time instants of $T = 10$ seconds, and that the future events depends only by the actual state of the system. It follows that the buffer state can be modelled as a DT-MC.

The transition graph of the system is provided below whereas its transition probability matrix



is

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ (1-p)(1-q) & q+p(1-q) \end{pmatrix} = \begin{pmatrix} 0.1 & 0.9 \\ 0.08 & 0.92 \end{pmatrix}$$

Let us now evaluate the ergodicity of this DT-MC by means of the Eigenvalue criteria. By solving

$$\det(\mathrm{sI} - \mathrm{P}) = \det \begin{pmatrix} s-0.1 & -0.9 \\ -0.08 & s-0.92 \end{pmatrix} = \mathrm{s}^2 - \frac{50}{51}\mathrm{s} + \frac{1}{50} = 0 \rightarrow \{\mathrm{s}_1 = 1, \mathrm{s}_2 = 0.02\}.$$

Since there is only one eigenvalue sastifying $|\lambda_i| = 1$, then the DT-MC is ergodic. Thus, the limiting distribution can be evaluated by solving the following linear system

$$\begin{cases} \Pi_s = \Pi_s \cdot \boldsymbol{P} \\ \Pi_s \cdot \boldsymbol{1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,0} = 0.1\pi_{s,0} + 0.08\pi_{s,1} \\ \pi_{s,1} = 0.9\pi_{s,0} + 0.92\pi_{s,1} \\ \pi_{s,0} + \pi_{s,1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,0} = \frac{0.08}{1-0.1}\pi_{s,1} \\ \pi_{s,1} = \frac{1}{1+\frac{0.08}{1-0.1}} \end{cases} \tag{7}$$

from which, it results that

$$\Pi_\ell \equiv \Pi_s \approx (0.0816, 0.9184)\,.$$

Let us not calculates the rate of discharged pieces η_d at steady state, that is the ratio between the probability that a piece is discharged, (i.e. the joint probability that $X_\infty = 1$, and, in the same time interval the robot fails its task), and the probability that a piece is introduced in the system (i.e. the probability of being in $X_\infty = 0$ and an incoming piece arrive plus the probability of being in $X_\tau = 1$ the robot terminates correctly its task, while in the same time interval a piece arrives). It follow that

$$\eta_d = \frac{\Pr(\text{a piece is discharged})}{\Pr(\text{a piece is intruduced in the system})} = \frac{pq \cdot \pi_{\ell,1}}{p \cdot \pi_{\ell,0} + p(1-q)\pi_{\ell,1}} \cdot 100 = 22.5,$$

where $\pi_{\ell,i} = \pi_i(\infty) = \Pr(X_\infty = i)$, with $i = 0, 1$.

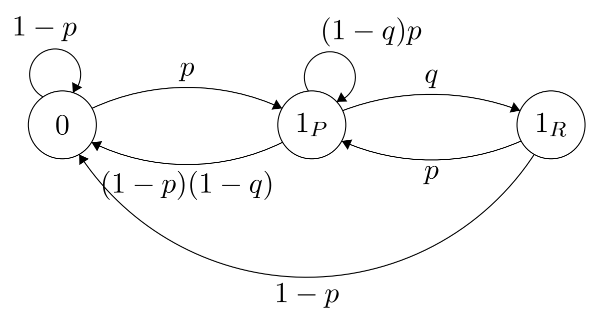
Let us now calculate the mean number of pieces in the system in the steady condition, that is denoted by $\bar{x}$. By invoking the definition of expected value we derive that

$$\bar{x} = \mathsf{E}[\#\text{ pieces in the system}] = \sum_{i=0}^{1} i \cdot \pi_i(\infty) = 0 \cdot \pi_{0,\ell} + 1 \cdot \pi_{1,\ell} = 0.9184 \text{ pieces.}$$

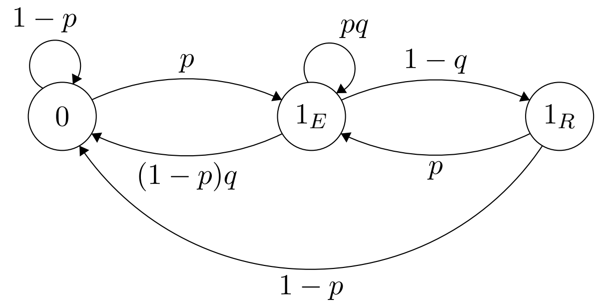
Finally, under the assumption that the robot may fail its task only once, it results that the now it is needed a way to count how many times the process is failed. To do that we can simply modify the DT-MC sample space to account the following events, resp.:

- the system is empty at time τ, denoted by "0";
- the piece in the system is under its first works, at time τ, denoted by "1_P";
- the piece in the system is under its second works, at time τ, denoted by 1_R.

Thus, by analogous considerations of that made at point (a) the DT-MC that describes this process can modified in accordance with the following transition graph



Solution of Exercise 7. The probability that a message sent by T_t is received to T_r is $p = 0.8$, whereas the probability that T_r correctly process the received message in a time interval of Δ [sec] is. $q = 0.6$. These two events are clearly independents. Under the assumption that the state space of the T_r station at time τ could be: Empty ($X_\tau = 0$); Elaborating a message ($X_\tau = 1_E$); Re-elaborating a message ($X_\tau = 1_R$) the following DT-MC take place



From that we can derive its corresponding transition probability matrix, that is

$$\boldsymbol{P} = \begin{pmatrix} p_{00} & p_{01} & 0 \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.12 & 0.48 & 0.4 \\ 0.2 & 0.8 & 0 \end{pmatrix} = \begin{pmatrix} 1-p & p & 0 \\ (1-p)q & pq & 1-q \\ 1-p & p & 0 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0.12 & 0.48 & 0.4 \\ 0.2 & 0.8 & 0 \end{pmatrix}$$

By inspection, note that the outgoing edges from 1_R consider only the case that a new message is arrived, i.e.,

$$p_{21} = \Pr(X_\tau = 1_E | X_\tau = 1_R) = p$$

or no message further messages are received, i.e.,

$$p_{20} = \Pr(X_\tau = 0 | X_\tau = 1_R) = 1 - p$$

because of the second elaboration always terminates correctly.

This DT-MC is ergodic because the graph is strongly connected, and thus its limiting probability distribution can be evaluated as its stationary distribution by solving the following linear system

$$\begin{cases} \Pi_s = \Pi_s \cdot \boldsymbol{P} \\ \Pi_s \cdot \mathbf{1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,1} = 0.2\pi_{s,1} + 0.12\pi_{s,2} + 0.2\pi_{s,3} \\ \pi_{s,2} = 0.8\pi_{s,1} + 0.48\pi_{s,2} + 0.8\pi_{s,3} \\ \pi_{s,3} = 0.4\pi_{s,2} \\ \pi_{s,1} + \pi_{s,2} + \pi_{s,3} = 1 \end{cases} \tag{8}$$

it results that

$$\Pi_\ell \equiv \Pi_s \approx (0.1515, 0.6061, 0.2424)\,.$$

Let us now calculate the mean number of pieces in the system at the steady-state. Let us first note our DT-MC describes the probability of having 0 messages in the system through π_{ℓ,x_0} or 1 pieces into the system through the probabilities of being either in x_1 or x_2. Thus the number of pieces in the system can be seen as random variable x which sample space is $X = \{0, 1\}$, and which probability mass function is $p_0 = \pi_{\ell,0}$ and $p_1 = \pi_{s,1_E} + \pi_{s,1_R}$. Thus the average number of pieces in the system in the steady condition corresponds to

$$\bar{x} = \mathsf{E}[x] = \sum_0^1 i \cdot p_i = 0 \cdot \pi_{\ell,0} + 1 \cdot (\pi_{\ell,1_E} + \pi_{\ell,1_R})$$

$$= 0 \cdot 0.1515 + 1 \cdot (0.6061 + 0.2424) = 0.8485 \text{ messages.}$$

Because of messages arrives every $\Delta = 20$ seconds, the rate of correctly elaborated messages at the steady state, denoted here by η for simplicity in notation, can expressed as the averaged number of correctly elaborated messages, with respect to the time unit Δ. In a time cycle no more that 1 message can stay within the system, thus the number of correctly elaborated messages can be described by random variable z which may takes only two values 0, or 1, and thus one as that $Z = 0, 1$. The probability mass function associated with $z = 1$, is

$$q_1 = q \cdot \pi_{\ell,1_E} + \pi_{\ell,1_R},$$

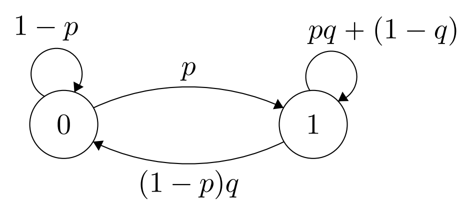
namely, the probability of being in 1_R (where the process always terminate correctly) plus the probability of remain in 1_E given that an incoming message is accepted to the system. On the other hand $q_0 = 1 - q_1$. It follows that

$$\mathsf{E}[\text{correctly elaborated messages}] = \mathsf{E}[z] = 0 \cdot q_0 + 1 \cdot q_1 = q \cdot \pi_{\ell,1_E} + \pi_{\ell,1_R} = 0.60606 \text{ messages}$$

and thus

$$\eta = \frac{\mathsf{E}[\text{correctly elaborated messages}]}{\Delta} = \frac{0.60606}{20} \approx 0.0303 \, \frac{\text{messages}}{\text{sec}}$$

Now, under the assumption that also the second elaboration could fail, the resulting stochastic process is simplified, by counting only the number of messages in T_r, as follows where it is



evident that

$$p_{11} = \Pr(X_\tau = 1 | X_{\tau-1} = 1) = pq + (1-q) \cdot p + (1-q)(1-p) = pq + (1-q),$$

namely, it is the sum of the probability that a new message is received and the current elaboration terminate correctly, plus the probability that elaboration fails (independently by the fact that a new message is arrived or not). From that modelling it result that

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ (1-p)q & pq + (1-q) \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 \\ 0.12 & 0.88 \end{pmatrix}$$

The DT-MC is ergodic because its graph is strongly connected and aperiodic.

To evaluated the rate of correctly elaborated messages at the steady state, we need to evaluate η defined as follows

$$\eta = \frac{\mathsf{E}[\text{correctly elaborated messages}]}{\Delta} = \frac{0 \cdot p_0 + 1 \cdot p_1}{\Delta}$$

where $p_1 = q\pi_{\ell,1}$ corresponds to the probability of staying in state 1 because a message has been correctly elaborated.

Since the process is ergodic it results that $\pi_{\ell,1}$ can be evaluated as the second component of its stationary distribution. Thus, we have that

$$\begin{cases} \Pi_s = \Pi_s \cdot \boldsymbol{P} \\ \Pi_s \cdot \mathbf{1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,x_0} = 0.2\pi_{s,x_0} + 0.12\pi_{s,x_1} \\ \pi_{s,x_1} = 0.8\pi_{s,x_0} + 0.88\pi_{s,x_1} \\ \pi_{s,x_0} + \pi_{s,x_1} = 1 \end{cases} \rightarrow \begin{cases} \pi_{s,x_0} = \frac{0.12}{1-0.2}\pi_{s,1} \\ \pi_{s,x_1} = \frac{1}{\frac{0.12}{1-0.2}+1} \end{cases} \tag{9}$$

from which, it results that

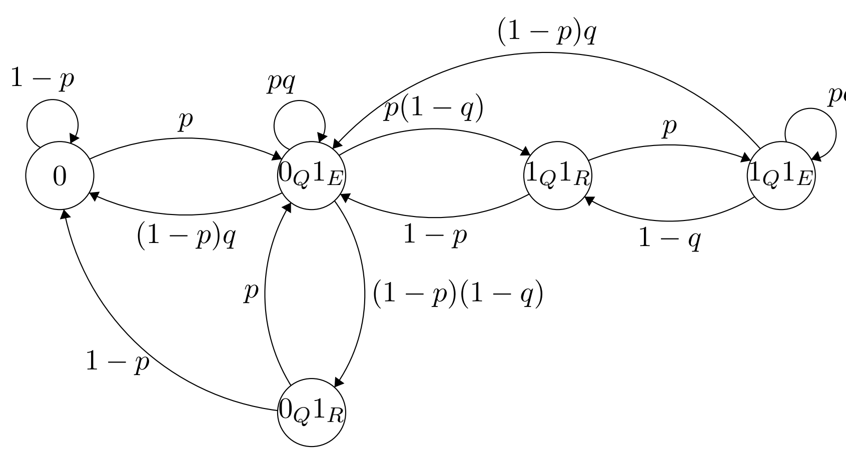
$$\Pi_\ell \equiv \Pi_s \approx (0.1304, 0.8696)\,.$$

Thus we have that

$$\eta = \frac{q \cdot \pi_{\ell,x_1}}{\Delta} = \frac{0 \cdot p_0 + 1 \cdot p_1}{\Delta} = \frac{0.52176}{20} = 0.0261\ \frac{\text{messages}}{\text{sec}}$$

Clearly, since now the processing task may fail more that one time, the rate of correctly elaborated messages is decreased.

Finally, if we assume that the T_r station has buffer size of 2 messagges; that i) only one message can be elaborated; ii) the second elaboration always terminates correctly, and by remanding the hint to consider a 5 states DT-MC, then, system can be modelled as follows where



- State 0 : correspond to 0 messages in the system;
- State 0_Q1_E : correspond to 1 message in the system at its first elaboration and 0 in the waiting line;
- State 0_Q1_R : correspond to 1 message in the system at its second elaboration and 0 in the waiting line;
- State 1_Q1_E : correspond to 2 message in the system, one in waiting line and one at its first elaboration;
- State 1_Q1_R : correspond to 2 message in the system, one in waiting line and one at its second elaboration;

Note that, since the second elaboration always terminate correctly, then from the state 1_Q1_R we only move to 0_Q1_E with probability equal to $1-p$, which means that a message in the wainting line is moved to the service system or to 1_Q1_E which means that a message in the wainting line is moved to the service system and another message with probability p is arrived.

Analogous considerations can be also made for the outgoing edges of each state to another.