Stochastic Models

A.A. 2019/2020

Assignment 5

Exercise 1. Draw the transition graph of the Continuous-time Markov Chain (CT-MC) which states space is $X = \{x_1, x_2, x_3\}$ and which transition rate matrix is

$$\boldsymbol{Q} = \begin{bmatrix} -10 & 10 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & -3 \end{bmatrix}.$$

Determine through the eigenvalue criteria if this CT-MC is ergodic.

Calculate its stationary distribution.

Exercise 2. Glaucoma is a disease of the eyes that usually occurs in old aged patients. Its progression, in the absence of treatment can be modelled as a time-homogeneous CT-MC which states are: mild ($x(t) = 1$), severe ($x(t) = 2$) and near blindness ($x(t) = 3$).

Direct progression from mild to sever blindness was never observed. However, the disease may evolve from mild to severe with a rate λ_1 and from severe to near blindness with a rate λ_2.

(a) Draw a transition graph that illustrates the progression of the disease, and write out the transition rate matrix $\boldsymbol{Q} \in \mathbb{R}^{3\times3}$.

(b) Let, at time $t = 0$, the first row of the transition probability matrix $\boldsymbol{P}(t) \in \mathbb{R}^{3\times3}$, be $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, use $\boldsymbol{Q}$ to derive the $p_{00}(t)$ and $p_{01}(t)$.

(c) Show that the average time taken to transition from state 0 to state 2 is given by

$$\mathsf{E}[T] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Exercise 3. Consider a continuous time homogenous and uniform birth-death process with three admissible states, resp., $X = \{0, 1, 2\}$. Let the birth and dead rates be denoted, resp., by λ and μ.

(a) Determine under which assumption the process is ergodic.

(b) Discuss under which assumptions its stationary distribution can be calculated.

Exercise 4. As single server machine in factory consists in a robot which task is to paint and polish car bodyworks. Car bodyworks arrive according to a Poisson process with parameters λ.

If the robot is busy the car bodyworks are diverted elsewhere (rejected). The time needed to paint a car bodywork or to polish are identical, independent, and and exponentially distributed. The mean service time is $1/\mu$ for each task. The painting process is modelled as a Bernoulli trial with parameter p. The polishing process always terminates with a success. Consider the next scenarios:

- S1: The painting process is repeated until it terminates correctly, then the car bodywork is polished.
- S2: The painting process is performed at most twice. If it is terminated correctly the car bodywork is then polished, otherwise it is rejected.
- S3: The painting process is performed at most twice. The second painting process always terminate correctly. Thus, after that, the car bodywork is polished.

(a) For each of the three scenarios, draw the transition graph and calculate its transition rate matrices.

(b) For each of the three scenarios, and under the assumption that the system operating at the steady regime, calculate the probability that an incoming car bodywork is rejected.

(c) For the second scenario, calculate the probability to reject a car bodywork after having failed its painting processes twice.

Exercise 5. An inventory-production system consists of a warehouse with capacity equal to 3 pieces and a conveyor belt that moves pieces from the depot and to a collection center.

The cart, in average, arrives empty at the warehouse 10 times per day according to a Poisson process. On average, the cart may load 3 pieces with probability 0.2, or 2 pieces with probability 0.5, or 1 piece with probability 0.3.

If the cart arrives at the warehouse and does not find any piece to be picked up, it returns back empty to the collection center. The warehouse is completely replenished with a $(s, S) = (1, 3)$ logistic policy, namely, once the stock becomes smaller or equal than $s = 1$, the warehouse state is restored to the inventory to a target $S = 3$. This event, occurs after a time that is exponentially distributed with a rate of 1/50 days.

(a) Describe the stochastic processes associated with the warehouse's states.

(b) Calculate the probability that, at the steady regime, the warehouse is empty and that of having the warehouse full.

(c) Calculate the mean number of pieces in the warehouse at steady state.

Exercise 6. The production system of a factory consists of a service station with 2 servers denoted by M_1 and M_2 and a waiting room. The system capacity is of 3 pieces and it counts also the pieces under works. The process of arrivals is Poisson with rate $\lambda = 10$. The service times of the two servers is exponentially distributed with rate μ_1 and μ_2.

The pieces, once arrived at the service station may be either be processed and then they leave the system, or wait in the buffer if there is place. Otherwise they are rejected.

For the following two use-cases provide a CT-MC modellization of the stochastic process associated with the state service station.

(a) Regular splitting: If the servers are both in idle, the next piece is forwarded to M_1.

(b) Random splitting: If the servers are both in idle, the next piece is forwarded either to M_1 or to M_2 with the same probability.

(c) Assuming that the two servers have the same service rate, namely, $\mu_1 = \mu_2 = 10$, for the two use-cases calculate the average number of pieces within the system.