Stochastic Models

A.A. 2019/2020

Assignement 3

Exercise 1. Consider 20 transmission sources, each transmits at a peak-rate of 10 Mb/s with probability $p_1 = 0.1$, or it is in idle mode. There are other 80 sources, each transmits at a peak-rate of 1 Mb/s with probability $p_2 = 0.05$ or it is in idle mode.

A service provider aims to allocate the minimal capacity C_{opt} such that no more than 0.0015 of the time, the demand of all these 100 sources exceeds the available capacity. Evaluate C_{opt}.

Exercise 2. Let us consider the following stochastic process: Roll a dice at time $\tau = 0$. Then, for each $\tau > 0$, if we get 2, 3, 4 o 5 at time τ, the dice is rolled again at time $\tau + 1$; if we get 1 or 6 the dice is left in the the previous position for the remaining time.

(a) Describe the stochastic process $\{X_\tau, \tau \in \mathbb{N}^+\}$ and derives the marginal probabilities $\pi_i(\tau)$, $\forall\ \tau$, and $i = 1, 2, 3, 4, 5, 6$

(b) Calculates the mean of this stochastic process, namely, $\mu_X(\tau) = \mathsf{E}[X_\tau]$, $\forall\ \tau \geqslant 0$, then derives $\mu_X(\infty) = \mathsf{E}[X_\tau]$

(c) Provides 3 possible realizations of this stochastic process for $\tau = 0, 1, 2, \ldots, 9$

(d) Calculate the following Joint probabilities, resp., $\Pr(X_0 = 2, X_1 = 6, X_2 = 6)$ and $\Pr(X_0 = 1, X_1 = 6, X_1 = 6)$.

(e) Discuss if this process is, respectively, stationary, strictly stationary, ergodic, independent, markovian.

Exercise 3. Let Z_1, Z_2, ..., be a family of independent and identically distributed random variables with probabilities $\Pr(Z_n = 1) = p$ e $\Pr(Z_n = -1) = 1 - p$ for each $n = 1, 2, \ldots$. Let us further define the random process

$$\{X_n, n = 0, 1, 2, \ldots\} \ : \ X_0 = 0, \qquad X_n = \sum_{k=1}^{n} Z_k, \qquad n = 1, 2, \ldots$$

The resulting stochastic process is also called *1-D random walk.*

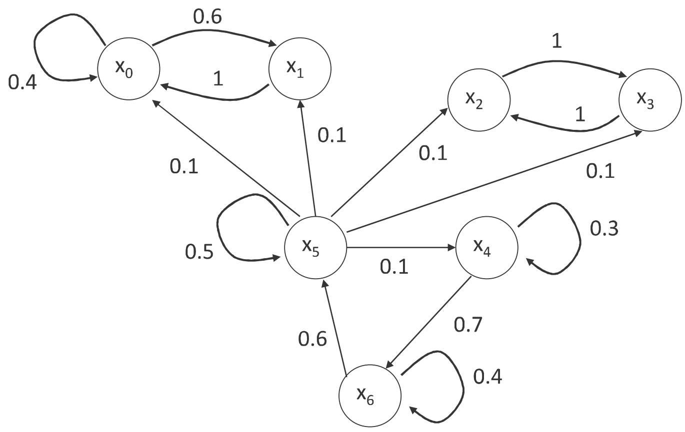
(a) Is $\{X_n,\ n \in \mathbb{N}\}$ a counting process?

(b) Is $\{X_n,\ n \in \mathbb{N}\}$ a markovian process?

(c) Provides a possible realization of $\{X_n,\ n \in \mathbb{N}\}$ for the first 10 instants of time

(d) Let $p = 0.5$, for the given the random walk, calculate $\mathsf{E}[X_n]$ and $\mathsf{Var}[X_n]$.

Exercise 4. Consider the Discrete-Time Markov Chain (DT-MC) depicted in the next figure.



(a) Evaluate the strongly connected components. Which are ergodics and which transients?

(b) Evaluate the periodicity of the ergodic components

(c) Evaluate if the given DT-MC is ergodic and explain what this property implies

(d) Evaluate if, by the change of the weights of some of the existing arc, the given analysis would change

(e) In the case that the reply to previous question is negative, evaluates if it is possible make the given DT-MC ergodic by simply changing the orientation of a single arc.

Exercise 5. Let $X = \{x_1, x_2, x_3\}$ be the vertex set associated to a given DT-MC which transition probability matrix is:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.9 & 0 \end{bmatrix}.$$

Calculate:

(a) the edge set $E \subset X \times X$ of the graph associated with the given DT-MC.

(b) if the given DT-MC is ergodic, or not, through both the grafical and analitic criteria.

(c) the stationary distribution of the given markov chain.

Exercise 6. In Exercise 6 of Assignment 2, it was provided a discussion, and the MatLab code, for generating an exponential random variable $Y \sim Exp(\lambda)$, by means of a uniform random variable $X \sim Uniform(0, 1)$, namely

```
n=1e4;                        % the length is arbitrary
lambda=40;                    % Inter-even time of Poisson 40 arrivals/sec
x=rand(n,1);                  % x~Uniform(0,1)
y = -lambda^-1*log(1-x);      % y~Exp(lambda)
mu_y=(1/n)*sum(y);            % Sampled mean of X (approx. E[X]=1/lambda)
```

(a) Reminding the meaning of the Erlang distribution, from `y` generate on MatLab a vector of occurrences `yE2` that is $Erlang(2, \lambda)$ distributed, where $\lambda = 40$.

(b) Reminding that for a Poisson process $\{Z_t, t \in T\}$, the inter-event time is $Y \sim Exp(\lambda)$. By means of `y` and the command `poissrnd` (see `help poissrnd`), generates on MatLab, a Poisson process which mean event rate is of 40arrivals/sec, and where the arrivals are scheduled by the inter-event random variable Y. Then verify if its sampled mean is equivalent to its expected value $\mathsf{E}[Z_t] = \lambda = 40\text{arrivals/sec}$.