Stochastic Models

A.A. 2019/2020

Assignement 2

Exercise 1. Let us consider a Poisson random variable X with parameter λ:

$$\Pr(X = i) = \frac{e^{-\lambda}\,\lambda^i}{i!}, \qquad i = 0, 1, 2, \ldots$$

(a) Proove that $\sum_{i=0}^{\infty} \Pr(X = i) = 1$.

(b) Calculate $\Pr(X > 2)$ with $\lambda = 4$.

Exercise 2. Consider a binary communication channel between a transmitter and a receiver where B_n is the value of the n-th bit at the receiver. Assume the probability that a bit is erroneous follows a Bernoulli distribution with parameter $p = 0.01$.

(a) Calculate the probability of having more than one error on a bit-stream of 10 bits.

(b) Show that a Poisson distribution with parameter λ can be used to approximate the Binomial distribution with parameter n e p iff $\lambda = np$, with n sufficiently big and p sufficiently small. Under this assumption, repeat the previous calculus.

Exercise 3. The random variable X counts the number of heads when we flip 3 coins. Calculate:

(a) the sample space of X

(b) the probability mass function $p_X(x)$

(c) the expected value $\mathsf{E}[X]$ and its variance $\mathsf{Var}[X]$ by means of the Moment Generating Function $\Pi_X(z) = \sum_{i=0}^{\infty} z^{-i} \cdot p_X(i)$.

Exercise 4. Let X_1 and X_2 be to independent and identical exponential random variables. Let $Y = X1 + X2$ be a new random variable. What is the probability density function of Y?

(a) What is the probability density function of Y?

(b) Where do we use the distribution of Y?

Exercise 5. Let X be a continuous random variable which probability density function follows

$$f_X(x) = \begin{cases} kx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}$ is a constant. Calculate

(a) the value of k in order to get a valid probability then evaluate its probability density function $f_X(x)$.

(b) the corresponding cumulative distribution function $F_X(x) = \Pr(X \leqslant x)$.

(c) the $\Pr(1/4 < X \leqslant 2)$.

(d) the expected value $\mathsf{E}[X]$.

(e) the variance $\mathsf{Var}[X] = \mathsf{E}[(X - \mathsf{E}[X])^2$.

Exercise 6. Let X be a continuous random variable. Let the random variable Y be defined as a function of X, such that $Y = F_Y^{-1}(X)$, that is $X = F_Y(Y)$, where $F_Y(y) = \Pr(Y \leqslant y)$ denotes an arbitrary cumulative probability distribution of Y.

(a) Prove that $F_Y(y) = 1 - F_X(x)$.

(b) Let $X \sim U(0,1)$ be uniformly distributed and $Y \sim Exp(\lambda)$ be exponential with parameter λ. Show how Y can be generated from X.

Exercise 7. Let $X \sim U(0,1)$ be an uniformly distributed random variable, with $0 \leqslant x \leqslant 1$. Let $Y = \sqrt{1 - X^2}$ be another random variable depending on X. Calculates:

(a) the expected values of X denoted by $\mathsf{E}[X]$

(b) the probability density function of Y, denoted by $f_Y(y)$

(c) the expected value of Y denoted by $\mathsf{E}[Y]$

Notice that $\int \sqrt{1 - x^2}\, dx = \frac{1}{2}\left(x \cdot \sqrt{1 - x^2} + \operatorname{asin}(x)\right)$.